

# Rate of convergence for eigenfunctions of Aharonov-Bohm operators with a moving pole

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December 6, 2016

## Abstract

We study the behavior of eigenfunctions for magnetic Aharonov-Bohm operators with half-integer circulation and Dirichlet boundary conditions in a planar domain. We prove a sharp estimate for the rate of convergence of eigenfunctions as the pole moves in the interior of the domain.

**Keywords.** Magnetic Schrödinger operators, Aharonov-Bohm potential, convergence of eigenfunctions.

**MSC classification.** 35J10, 35Q40, 35J75.

## 1 Introduction

For every  $a = (a_1, a_2) \in \mathbb{R}^2$ , we consider the Aharonov-Bohm vector potential with pole  $a$  and circulation  $1/2$  defined as

$$A_a(x_1, x_2) = A_0(x_1 - a_1, x_2 - a_2), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\},$$

where

$$A_0(x_1, x_2) = \frac{1}{2} \left( \frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right), \quad (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

The Aharonov-Bohm vector potential  $A_a$  generates a  $\delta$ -type magnetic field, which is called Aharonov-Bohm field: this field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane  $(x_1, x_2)$  at the point  $a$ , as the radius of the solenoid tends to zero while the flux through the solenoid section remains constantly equal to  $1/2$ . Neglecting the irrelevant coordinate along the solenoid axis, the problem becomes 2-dimensional.

Let  $\Omega \subset \mathbb{R}^2$  be a bounded, open and simply connected domain. For every  $a \in \Omega$ , we consider the eigenvalue problem

$$\begin{cases} (i\nabla + A_a)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (E_a)$$

in a weak sense, where the magnetic Schrödinger operator with Aharonov-Bohm vector potential  $(i\nabla + A_a)^2$  acts on functions  $u : \mathbb{R}^2 \rightarrow \mathbb{C}$  as

$$(i\nabla + A_a)^2 u = -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u.$$

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A suitable functional setting for stating a weak formulation of  $(E_a)$  can be introduced as follows: for every  $a \in \Omega$ , the functional space  $H^{1,a}(\Omega, \mathbb{C})$  is defined as the completion of

$$\{u \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : u \text{ vanishes in a neighborhood of } a\}$$

with respect to the norm

$$\|u\|_{H^{1,a}(\Omega, \mathbb{C})} = \left( \|(i\nabla + A_a)u\|_{L^2(\Omega, \mathbb{C}^2)}^2 + \|u\|_{L^2(\Omega, \mathbb{C})}^2 \right)^{1/2}.$$

In view of the following Hardy type inequality proved in [12]

$$\int_{\mathbb{R}^2} |(i\nabla + A_a)u|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x-a|^2} dx,$$

which holds for all  $a \in \mathbb{R}^2$  and  $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{C})$ , it is easy to verify that

$$H^{1,a}(\Omega, \mathbb{C}) = \{u \in H^1(\Omega, \mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega, \mathbb{C})\}.$$

We also denote as  $H_0^{1,a}(\Omega, \mathbb{C})$  the space obtained as the completion of  $C_c^\infty(\Omega \setminus \{a\}, \mathbb{C})$  with respect to the norm  $\|\cdot\|_{H^{1,a}(\Omega, \mathbb{C})}$ , so that  $H_0^{1,a}(\Omega, \mathbb{C}) = \{u \in H_0^1(\Omega, \mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega, \mathbb{C})\}$ .

For every  $a \in \Omega$ , we say that  $\lambda \in \mathbb{R}$  is an eigenvalue of problem  $(E_a)$  in a weak sense if there exists  $u \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$  (called an eigenfunction) such that

$$\int_{\Omega} (i\nabla u + A_a u) \cdot \overline{(i\nabla v + A_a v)} dx = \lambda \int_{\Omega} u \bar{v} dx \quad \text{for all } v \in H_0^{1,a}(\Omega, \mathbb{C}).$$

From classical spectral theory, the eigenvalue problem  $(E_a)$  admits a sequence of real diverging eigenvalues (repeated according to their finite multiplicity)  $\lambda_1^a \leq \lambda_2^a \leq \dots \leq \lambda_j^a \leq \dots$ .

The mathematical interest in Aharonov-Bohm operators with half-integer circulation can be motivated by a strong relation between spectral minimal partitions of the Dirichlet Laplacian with points of odd multiplicity and nodal domains of eigenfunctions of these operators. Indeed, a magnetic characterization of minimal partitions was given in [10] (see also [5, 6, 7, 15]): partitions with points of odd multiplicity can be obtained as nodal domains by minimizing a certain eigenvalue of an Aharonov-Bohm Hamiltonian with respect to the number and the position of poles. From this, a natural interest in the study of the properties of the map  $a \mapsto \lambda_j^a$  (associating eigenvalues of magnetic operators to the position of poles) arises. In [1, 2, 3, 8, 13, 14] the behaviour of the function  $a \mapsto \lambda_j^a$  in a neighborhood of a fixed point  $b \in \overline{\Omega}$  has been investigated, both in the cases  $b \in \Omega$  and  $b \in \partial\Omega$ . In particular, the analysis carried out in [1, 2, 3, 8, 14] shows that, as the pole moves towards a fixed limit pole  $b \in \overline{\Omega}$ , the rate of convergence of  $\lambda_j^a$  to  $\lambda_j^b$  is related to the number of nodal lines of the limit eigenfunction meeting at  $b$ . In the present paper we aim at deepening this analysis describing also the behaviour of the corresponding eigenfunctions; in particular, we will derive a sharp estimate for the rate of convergence of eigenfunctions associated to moving poles, in terms of the number of nodal lines of the limit eigenfunction.

Without loss of generality, we can assume that

$$b = 0 \in \Omega.$$

Let us assume that there exists  $n_0 \geq 1$  such that

$$\lambda_{n_0}^0 \text{ is simple,} \tag{1}$$

and denote  $\lambda_0 = \lambda_{n_0}^0$  and, for any  $a \in \Omega$ ,  $\lambda_a = \lambda_{n_0}^a$ . From [13, Theorem 1.3] it follows that the map  $a \mapsto \lambda_a$  is analytic in a neighborhood of 0; in particular we have that

$$\lambda_a \rightarrow \lambda_0, \quad \text{as } a \rightarrow 0. \tag{2}$$

Let  $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C}) \setminus \{0\}$  be a  $L^2(\Omega, \mathbb{C})$ -normalized eigenfunction of problem  $(E_0)$  associated to the eigenvalue  $\lambda_0 = \lambda_{n_0}^0$ , i.e. satisfying

$$\begin{cases} (i\nabla + A_0)^2 \varphi_0 = \lambda_0 \varphi_0, & \text{in } \Omega, \\ \varphi_0 = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} |\varphi_0(x)|^2 dx = 1. \end{cases} \quad (3)$$

From [9, Theorem 1.3] (see also [15, Theorem 1.5]) it is known that  $\varphi_0$  has at 0 a zero of order  $\frac{k}{2}$  for some odd  $k \in \mathbb{N}$ , i.e. there exist  $k \in \mathbb{N}$  odd and  $\beta_1, \beta_2 \in \mathbb{C}$  such that  $(\beta_1, \beta_2) \neq (0, 0)$  and

$$r^{-k/2} \varphi_0(r(\cos t, \sin t)) \rightarrow e^{i\frac{k}{2}} \left( \beta_1 \cos\left(\frac{k}{2}t\right) + \beta_2 \sin\left(\frac{k}{2}t\right) \right) \quad \text{in } C^{1,\tau}([0, 2\pi], \mathbb{C}) \quad (4)$$

as  $r \rightarrow 0^+$  for any  $\tau \in (0, 1)$ . The asymptotics (4) (together with the fact that the right hand side of (4) is a complex multiple of a real-valued function, see [11]) implies that  $\varphi_0$  has exactly  $k$  nodal lines meeting at 0 and dividing the whole angle into  $k$  equal parts; such nodal lines are tangent to the  $k$  half-lines

$$\left\{ \left( t, \tan\left(\alpha_0 + j\frac{2\pi}{k}\right)t \right) : t > 0 \right\}, \quad j = 0, 1, \dots, k-1,$$

for some angle  $\alpha_0 \in [0, \frac{2\pi}{k})$ .

In [1, 2] it has been proved that, under assumption (1) and being  $k$  as in (4),

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \rightarrow C_0 \cos(k(\alpha - \alpha_0)) \quad \text{as } a \rightarrow 0 \text{ with } a = |a|(\cos \alpha, \sin \alpha), \quad (5)$$

where  $C_0 > 0$  is a positive constant depending only on  $k$ ,  $\beta_1$ , and  $\beta_2$ . More precisely, in [1, 2] it has been proved that

$$C_0 = -4(|\beta_1|^2 + |\beta_2|^2) \mathfrak{m}_k$$

where

$$\mathfrak{m}_k = \min_{u \in \mathcal{D}_s^{1,2}(\mathbb{R}_+^2)} \left[ \frac{1}{2} \int_{\mathbb{R}_+^2} |\nabla u(x)|^2 dx - \frac{k}{2} \int_0^1 t^{\frac{k}{2}-1} u(t, 0) dt \right] < 0. \quad (6)$$

In (6),  $s$  denotes the half-line  $s := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geq 1\}$  and  $\mathcal{D}_s^{1,2}(\mathbb{R}_+^2)$  is the completion of  $C_c^\infty(\overline{\mathbb{R}_+^2} \setminus s)$  under the norm  $(\int_{\mathbb{R}_+^2} |\nabla u|^2 dx)^{1/2}$ .

Let us now consider a suitable family of eigenfunctions relative to the approximating eigenvalue  $\lambda_a$ . In order to choose eigenfunctions with a suitably normalized phase, let us introduce the following notations. For every  $\alpha \in [0, 2\pi)$  and  $b = (b_1, b_2) = |b|(\cos \alpha, \sin \alpha) \in \mathbb{R}^2 \setminus \{0\}$ , we define

$$\theta_b : \mathbb{R}^2 \setminus \{b\} \rightarrow [\alpha, \alpha + 2\pi) \quad \text{and} \quad \theta_0^b : \mathbb{R}^2 \setminus \{0\} \rightarrow [\alpha, \alpha + 2\pi)$$

such that

$$\theta_b(b + r(\cos t, \sin t)) = t \quad \text{and} \quad \theta_0^b(r(\cos t, \sin t)) = t, \quad \text{for all } r > 0 \text{ and } t \in [\alpha, \alpha + 2\pi).$$

We also define

$$\theta_0 : \mathbb{R}^2 \setminus \{0\} \rightarrow [0, 2\pi)$$

such that

$$\theta_0(\cos t, \sin t) = t \quad \text{for all } t \in [0, 2\pi).$$

For all  $a \in \Omega$ , let  $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$  be an eigenfunction of problem  $(E_a)$  associated to the eigenvalue  $\lambda_a$ , i.e. solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_a = \lambda_a \varphi_a, & \text{in } \Omega, \\ \varphi_a = 0, & \text{on } \partial\Omega, \end{cases} \quad (7)$$

such that its modulus and phase are normalized in such a way that

$$\int_{\Omega} |\varphi_a(x)|^2 dx = 1 \quad \text{and} \quad \int_{\Omega} e^{\frac{i}{2}(\theta_0^a - \theta_a)(x)} \varphi_a(x) \overline{\varphi_0(x)} dx \text{ is a positive real number,} \quad (8)$$

where  $\varphi_0$  is as in (3). From (1), (2), (3), (7), (8), and standard elliptic estimates, it follows that  $\varphi_a \rightarrow \varphi_0$  in  $H^1(\Omega, \mathbb{C})$  and in  $C_{\text{loc}}^2(\Omega \setminus \{0\}, \mathbb{C})$  and

$$(i\nabla + A_a)\varphi_a \rightarrow (i\nabla + A_0)\varphi_0 \quad \text{in } L^2(\Omega, \mathbb{C}). \quad (9)$$

The main result of the present paper establishes the sharp rate of the convergence (9).

**Theorem 1.1.** *For  $\alpha \in \mathbb{R}$ ,  $p = (\cos \alpha, \sin \alpha)$  and  $a = |a|p \in \Omega$ , let  $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$  solve equation (7-8) and  $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$  be a solution to (3) satisfying (1) and (4). Then there exists  $\mathfrak{L}_p > 0$  such that*

$$|a|^{-k} \left\| (i\nabla + A_a)\varphi_a - e^{\frac{i}{2}(\theta_a - \theta_0^a)}(i\nabla + A_0)\varphi_0 \right\|_{L^2(\Omega, \mathbb{C})}^2 \rightarrow (|\beta_1|^2 + |\beta_2|^2) \mathfrak{L}_p \quad (10)$$

as  $a = |a|p \rightarrow 0$ . Moreover the function  $\alpha \mapsto \mathfrak{L}_{(\cos \alpha, \sin \alpha)}$  is continuous, even, and periodic with period  $\frac{2\pi}{k}$ .

The constant  $\mathfrak{L}_p$  in Theorem 1.1 can be characterized as the energy of the solution of an elliptic problem with cracks (see (22)), where jumping conditions are prescribed on the segment connecting 0 and  $p$  and on the tangent to a nodal line of  $\varphi_0$ , see section 3.

For every  $\alpha \in \mathbb{R}$ , let us denote as  $s_\alpha = \{t(\cos \alpha, \sin \alpha) : t \geq 0\}$  the half-line with slope  $\alpha$ . We notice that, if  $a = |a|(\cos \alpha, \sin \alpha)$ , then  $\nabla(\frac{\theta_a}{2}) = A_a$ ,  $\nabla(\frac{\theta_0^a}{2}) = A_0$ , and  $e^{-\frac{i}{2}\theta_a}$  and  $e^{-\frac{i}{2}\theta_0^a}$  are smooth in  $\Omega \setminus s_\alpha$ . Thus

$$i\nabla_{\Omega \setminus s_\alpha}(e^{-\frac{i}{2}\theta_a}\varphi_a) = e^{-\frac{i}{2}\theta_a}(i\nabla + A_a)\varphi_a, \quad i\nabla_{\Omega \setminus s_\alpha}(e^{-\frac{i}{2}\theta_0^a}\varphi_0) = e^{-\frac{i}{2}\theta_0^a}(i\nabla + A_0)\varphi_0,$$

where  $\nabla_{\Omega \setminus s_\alpha}$  is the distributional gradient in  $\Omega \setminus s_\alpha$ . Hence (10) can be rewritten as

$$|a|^{-k} \left\| \nabla_{\Omega \setminus s_\alpha}(e^{-\frac{i}{2}\theta_a}\varphi_a - e^{-\frac{i}{2}\theta_0^a}\varphi_0) \right\|_{L^2(\Omega, \mathbb{C})}^2 \rightarrow (|\beta_1|^2 + |\beta_2|^2) \mathfrak{L}_p$$

as  $a = |a|p \rightarrow 0$ ; thus it can be interpreted as a sharp asymptotics of the rate of convergence of the approximating eigenfunction to the limit eigenfunction in the space  $\{u \in H^1(\Omega \setminus s_\alpha) : u = 0 \text{ on } \partial\Omega\}$ .

The paper is organized as follows. In section 2 we fix some notation and recall some known facts. In section 3 we give a variational characterization of the limit profile of scaled eigenfunctions, which is used to study the properties (positivity, evenness, periodicity) of the function  $p \mapsto \mathfrak{L}_p$ . Finally, in section 4 we prove Theorem 1.1, providing estimates of the energy variation first inside disks with radius  $R|a|$  and then outside such disks; this latter outer estimate is performed exploiting the invertibility of an operator associated to the limit eigenvalue problem. We mention that this strategy was first developed in [4] in the context of spectral stability for varying domains, obtained by adding thin handles to a fixed limit domain.

## 2 Preliminaries and some known facts

Through a rotation, we can easily choose a coordinate system in such a way that one nodal line of  $\varphi_0$  is tangent to the  $x_1$ -axis, i.e.  $\alpha_0 = 0$ . In this coordinate system, we have that, letting  $\beta_1, \beta_2$  be as in (4),

$$\beta_1 = 0. \quad (11)$$

The asymptotics of eigenvalues established in [1, 2], as well as the estimates for eigenfunctions we are going to achieve in the present paper, are based on a blow-up analysis for scaled eigenfunctions performed in [1, 2], whose main results are briefly recalled below for the sake of completeness.

For every  $p \in \mathbb{R}^2$  and  $r > 0$ , we denote as  $D_r(p)$  the disk of center  $p$  and radius  $r$  and as  $D_r = D_r(0)$  the disk of center 0 and radius  $r$ . Moreover we denote, for every  $r > 0$ ,  $D_r^+ = \{(x_1, x_2) \in D_r : x_2 > 0\}$  and  $D_r^- = \{(x_1, x_2) \in D_r : x_2 < 0\}$ .

First of all, we observe that (4) completely describes the behaviour of  $\varphi_0$  after scaling; indeed, letting

$$W_a(x) := \frac{\varphi_0(|a|x)}{|a|^{k/2}},$$

from [9, Theorem 1.3 and Lemma 6.1] we have that, under condition (11),

$$W_a \rightarrow \beta_2 e^{\frac{i}{2}\theta_0} \psi \quad \text{as } |a| \rightarrow 0 \quad (12)$$

in  $H^{1,0}(D_R, \mathbb{C})$  for every  $R > 1$ , where  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  is the  $\frac{k}{2}$ -homogeneous function (which is harmonic on  $\mathbb{R}^2 \setminus \{(r, 0) : r \geq 0\}$ )

$$\psi(r \cos t, r \sin t) = r^{k/2} \sin\left(\frac{k}{2}t\right), \quad r \geq 0, \quad t \in [0, 2\pi]. \quad (13)$$

For every  $p \in \mathbb{R}^2$ , we denote by  $\mathcal{D}_p^{1,2}(\mathbb{R}^2, \mathbb{C})$  the completion of  $C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$  with respect to the magnetic Dirichlet norm

$$\|u\|_{\mathcal{D}_p^{1,2}(\mathbb{R}^2, \mathbb{C})} := \left( \int_{\mathbb{R}^2} |(i\nabla + A_p)u(x)|^2 dx \right)^{1/2}. \quad (14)$$

**Proposition 2.1** ([2], Proposition 4). *Let  $\alpha \in [0, 2\pi)$  and  $p = (\cos \alpha, \sin \alpha)$ . There exists a unique function  $\Psi_p \in H_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathbb{C})$  such that*

$$(i\nabla + A_p)^2 \Psi_p = 0 \quad \text{in } \mathbb{R}^2 \text{ in a weak } H^{1,p}\text{-sense,} \quad (15)$$

and

$$\int_{\mathbb{R}^2 \setminus D_r} |(i\nabla + A_p)(\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi)|^2 dx < +\infty, \quad \text{for any } r > 1, \quad (16)$$

where  $\psi$  is defined in (13). Furthermore (see [9, Theorem 1.5])

$$\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi = O(|x|^{-1/2}), \quad \text{as } |x| \rightarrow +\infty.$$

**Theorem 2.2** ([2], Theorem 11 and Remark 12). *For  $\alpha \in [0, 2\pi)$ ,  $p = (\cos \alpha, \sin \alpha)$  and  $a = |a|p \in \Omega$ , let  $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$  solve (7-8) and  $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$  be a solution to (3) satisfying (1), (4), and (11). Let  $\Psi_p$  be as in Proposition 2.1. Then*

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \rightarrow \beta_2 \Psi_p \quad \text{as } a = |a|p \rightarrow 0,$$

in  $H^{1,p}(D_R, \mathbb{C})$  for every  $R > 1$  and in  $C_{\text{loc}}^2(\mathbb{R}^2 \setminus \{p\}, \mathbb{C})$ .

In the sequel, we will denote

$$\tilde{\varphi}_a(x) = \frac{\varphi_a(|a|x)}{|a|^{k/2}}.$$

Sharp estimates of the energy variation under moving of poles will be derived by approximating the eigenfunction  $\varphi_a$  by  $H^{1,0}$ -functions in the less expensive way from the energetic point of view. For every  $R > 2$  and  $|a|$  sufficiently small, we define these approximating functions  $v_{R,a}$  as follows:

$$v_{R,a} = \begin{cases} v_{R,a}^{ext}, & \text{in } \Omega \setminus D_{R|a|}, \\ v_{R,a}^{int}, & \text{in } D_{R|a|}, \end{cases}$$

where

$$v_{R,a}^{ext} := e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_a \quad \text{in } \Omega \setminus D_{R|a|}$$

solves

$$\begin{cases} (i\nabla + A_0)^2 v_{R,a}^{ext} = \lambda_a v_{R,a}^{ext}, & \text{in } \Omega \setminus D_{R|a|}, \\ v_{R,a}^{ext} = e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_a & \text{on } \partial(\Omega \setminus D_{R|a|}), \end{cases}$$

whereas  $v_{R,a}^{int}$  is the unique solution to the problem

$$\begin{cases} (i\nabla + A_0)^2 v_{R,a}^{int} = 0, & \text{in } D_{R|a|}, \\ v_{R,a}^{int} = e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_a, & \text{on } \partial D_{R|a|}. \end{cases}$$

We notice that  $v_{R,a} \in H_0^{1,0}(\Omega, \mathbb{C})$  for all  $R > 2$  and  $a$  sufficiently small. For all  $R > 2$  and  $a = |a|p \in \Omega$  with  $|a|$  small, we define

$$Z_a^R(x) := \frac{v_{R,a}^{int}(|a|x)}{|a|^{k/2}}. \quad (17)$$

For all  $R > 2$  and  $p = (\cos \alpha, \sin \alpha)$ , we also define  $z_{p,R}$  as the unique solution to

$$\begin{cases} (i\nabla + A_0)^2 z_{p,R} = 0, & \text{in } D_R, \\ z_{p,R} = e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p, & \text{on } \partial D_R, \end{cases} \quad (18)$$

with  $\Psi_p$  as in Proposition 2.1.

**Lemma 2.3** ([2], Remark 12; [1], Lemma 8.3). *For  $R > 2$ ,  $\alpha \in [0, 2\pi)$ ,  $p = (\cos \alpha, \sin \alpha)$  and  $a = |a|p \in \Omega$  small, let  $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$  solve (7-8),  $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$  be a solution to (3) satisfying (1), (4), and (11), and  $Z_a^R$  be as in (17). Then*

$$Z_a^R \rightarrow \beta_2 z_{p,R} \quad \text{as } a = |a|p \rightarrow 0 \text{ in } H^{1,0}(D_R, \mathbb{C}) \text{ for every } R > 2,$$

with  $z_{p,R}$  being as in (18).

### 3 Variational characterization of the limit profile $\Psi_p$

In [1], the limit profile  $\Psi_p$  was constructed by solving a minimization problem in the case  $p = (1, 0)$  (i.e. for poles moving tangentially to a nodal line of the limit eigenfunction); in that case the limit profile was null on a half-line. In the spirit of [3] (where poles moving towards the boundary were considered), we extend this variational construction for poles moving along a generic direction  $p = (\cos \alpha, \sin \alpha)$  and construct the limit profile by solving an elliptic crack problem prescribing the jump of the solution along the segment joining 0 and  $p$ .

Let us fix  $\alpha \in (0, 2\pi)$  and  $p = (\cos \alpha, \sin \alpha) \in \mathbb{S}^1$ . We denote by  $\Gamma_p$  the segment joining 0 to  $p$ , that is to say

$$\Gamma_p = \{(r \cos \alpha, r \sin \alpha) : r \in (0, 1)\}.$$

Let  $s_0 = \{(x_1, 0) : x_1 \geq 0\}$ . We introduce the trace operators

$$\gamma^\pm : \bigcap_{R>0} H^1(D_R^\pm \setminus \Gamma_p) \longrightarrow H_{\text{loc}}^{1/2}(s_0).$$

We also define  $\mathcal{H}$  as the completion of

$$\mathcal{D} = \{u \in H^1(\mathbb{R}^2 \setminus s_0) : \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0 \text{ and } u = 0 \text{ in neighborhoods of } 0 \text{ and } \infty\}$$

with respect to the Dirichlet norm  $(\int_{\mathbb{R}^2 \setminus s_0} |\nabla u|^2)^{1/2}$ . In the following lemma we prove that a Hardy-type inequality can be recovered even in dimension 2, under the jump condition  $\gamma^+(u) + \gamma^-(u) = 0$  forced for  $\mathcal{H}$ -functions.

**Lemma 3.1.** *The functions in  $\mathcal{D}$  satisfy the following Hardy-type inequality:*

$$\int_{\mathbb{R}^2 \setminus s_0} |\nabla \varphi(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|\varphi(x)|^2}{|x|^2} dx \quad \text{for all } u \in \mathcal{D}.$$

*Proof.* This is a consequence of a suitable change of gauge combined with the Hardy-type inequality for magnetic Sobolev spaces proved in [12]. For any  $\varphi \in \mathcal{D}$ , the function  $u := e^{\frac{i}{2}\theta_0}\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})$  according to the definition of the spaces  $\mathcal{D}_p^{1,2}(\mathbb{R}^2, \mathbb{C})$  given in Section 2 (see (14)). From the Hardy-type inequality proved in [12], it follows that

$$\int_{\mathbb{R}^2} |(i\nabla + A_0)u(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} dx.$$

Since  $\nabla(\frac{\theta_0}{2}) = A_0$  and  $(i\nabla + A_0)u = ie^{\frac{i}{2}\theta_0}\nabla\varphi$  in  $\mathbb{R}^2 \setminus s_0$ , we have that

$$\int_{\mathbb{R}^2} |(i\nabla + A_0)u(x)|^2 dx = \int_{\mathbb{R}^2 \setminus s_0} |\nabla \varphi(x)|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} dx = \int_{\mathbb{R}^2} \frac{|\varphi(x)|^2}{|x|^2} dx,$$

thus the proof is complete.  $\square$

As a direct consequence of Lemma 3.1,  $\mathcal{H}$  can be characterized as

$$\mathcal{H} = \left\{ u \in L_{\text{loc}}^1(\mathbb{R}^2) : \nabla_{\mathbb{R}^2 \setminus s_0} u \in L^2(\mathbb{R}^2), \frac{u}{|x|} \in L^2(\mathbb{R}^2), \text{ and } \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0 \right\},$$

where  $\nabla_{\mathbb{R}^2 \setminus s_0} u$  denotes the distributional gradient of  $u$  in  $\mathbb{R}^2 \setminus s_0$ .

For  $p \neq e$  with  $e = (1, 0)$ , we also define the space  $\mathcal{H}_p$  as the completion of

$$\mathcal{D}_p = \{ u \in H^1(\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)) : \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0 \text{ and } u = 0 \text{ in neighborhoods of } 0 \text{ and } \infty \}$$

with respect to the Dirichlet norm

$$\|u\|_{\mathcal{H}_p} := \|\nabla u\|_{L^2(\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p))}. \quad (19)$$

In order to prove that the space  $\mathcal{H}_p$  defined above is a concrete functional space, the argument performed in Lemma 3.1 is no more suitable, since  $\mathcal{H}_p$ -functions do not satisfy a Hardy inequality in the whole  $\mathbb{R}^2$ . We need the following two lemmas, which establish a Hardy inequality in external domains and a Poincaré inequality in  $D_1$  for  $\mathcal{H}_p$ -functions.

**Lemma 3.2.** *The functions in  $\mathcal{H}_p$  satisfy the following Hardy-type inequality in  $\mathbb{R}^2 \setminus D_1$ :*

$$\|\varphi\|_{\mathcal{H}_p}^2 \geq \frac{1}{4} \int_{\mathbb{R}^2 \setminus D_1} \frac{|\varphi(x)|^2}{|x|^2} dx, \quad \text{for all } \varphi \in \mathcal{H}_p.$$

*Proof.* The proof follows via a change of gauge as in the proof of Lemma 3.1. More precisely, we notice that, for any  $\varphi \in \mathcal{D}_p$ , the function  $u$  defined as  $u = e^{\frac{i}{2}\theta_0}\varphi$  in  $\mathbb{R}^2 \setminus D_1$  and as  $u(x) = u(x)/|x|^2$  in  $D_1$  belongs to  $\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})$ . From the invariance of Dirichlet magnetic norms and Hardy norms by Kelvin transform and the Hardy-type inequality of [12], it follows that

$$\begin{aligned} \|\varphi\|_{\mathcal{H}_p}^2 &\geq \int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} |\nabla \varphi(x)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |(i\nabla + A_0)u(x)|^2 dx \\ &\geq \frac{1}{8} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2} dx = \frac{1}{4} \int_{\mathbb{R}^2 \setminus D_1} \frac{|\varphi(x)|^2}{|x|^2} dx. \end{aligned}$$

The conclusion follows by density of  $\mathcal{D}_p$  in  $\mathcal{H}_p$ .  $\square$

**Lemma 3.3.** *The functions in  $\mathcal{H}_p$  satisfy the following Poincaré inequality in  $D_1$ :*

$$\|\varphi\|_{\mathcal{H}_p}^2 \geq \frac{1}{6} \int_{D_1} |\varphi(x)|^2 dx, \quad \text{for all } \varphi \in \mathcal{H}_p.$$

*Proof.* From the Divergence Theorem, the Schwarz inequality and the diamagnetic inequality, it follows that, for every  $u \in H^{1,0}(D_1 \setminus \Gamma_p)$ ,

$$\begin{aligned} 2 \int_{D_1} |u|^2 dx &= \int_{D_1 \setminus \Gamma_p} \left( \operatorname{div}(|u|^2 x) - 2|u| |\nabla u| \cdot x \right) dx \\ &\leq \int_{\partial D_1} |u|^2 ds + \int_{D_1 \setminus \Gamma_p} |u|^2 dx + \int_{D_1 \setminus \Gamma_p} |\nabla u|^2 dx \\ &\leq \int_{\partial D_1} |u|^2 ds + \int_{D_1} |u|^2 dx + \int_{D_1 \setminus \Gamma_p} |(i\nabla + A_0)u|^2 dx \end{aligned}$$

where, when applying the Divergence Theorem, we have use the fact that  $x \cdot \nu = 0$  on both sides of  $\Gamma_p$ . If  $\varphi \in \mathcal{D}_p$ , then  $u := e^{\frac{i}{2}\theta_0} \varphi \in H^{1,0}(D_1 \setminus \Gamma_p)$  and  $(i\nabla + A_0)u = ie^{\frac{i}{2}\theta_0} \nabla \varphi$  in  $D_1 \setminus (s_0 \cup \Gamma_p)$ , hence the previous inequality yields

$$\int_{D_1} |\varphi|^2 dx \leq \int_{\partial D_1} |\varphi|^2 ds + \int_{D_1 \setminus (s_0 \cup \Gamma_p)} |\nabla \varphi|^2 dx.$$

On the other hand, via the Divergence Theorem,

$$\begin{aligned} \int_{\partial D_1} |\varphi|^2 &= \int_{\partial D_1} \varphi^2 \frac{x}{|x|^2} \cdot \nu \\ &= - \int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} \operatorname{div} \left( \varphi^2 \frac{x}{|x|^2} \right) + \int_0^{+\infty} \gamma^+(\varphi^2) \frac{(s, 0)}{s^2} \cdot (0, -1) ds + \int_0^{+\infty} \gamma^-(\varphi^2) \frac{(s, 0)}{s^2} \cdot (0, 1) ds \\ &= - \int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} \operatorname{div} \left( \varphi^2 \frac{x}{|x|^2} \right) = -2 \int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} \varphi \nabla \varphi \cdot \frac{x}{|x|^2} \\ &\leq \int_{\mathbb{R}^2 \setminus (D_1 \cup s_0)} |\nabla \varphi|^2 + \int_{\mathbb{R}^2 \setminus D_1} \frac{|\varphi|^2}{|x|^2} \leq 5 \|\varphi\|_{\mathcal{H}_p}^2, \end{aligned}$$

where the last inequality is obtained by Lemma 3.2. The proof is thus complete.  $\square$

As a straightforward consequence of Lemma 3.2 and Lemma 3.3, we can characterize the space  $\mathcal{H}_p$  as

$$\left\{ u \in L_{\text{loc}}^1(\mathbb{R}^2) : \nabla_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} u \in L^2(\mathbb{R}^2), \frac{u}{|x|} \in L^2(\mathbb{R}^2 \setminus D_1), u \in L^2(D_1), \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s_0 \right\}.$$

The functions in  $\mathcal{H}_p$  may clearly be discontinuous on  $\Gamma_p$ . For this reason, we introduce two trace operators. Let us consider the sets  $U_p^+ = \{(x_1, x_2) \in \mathbb{R}^2 : \cos \alpha x_2 > \sin \alpha x_1\} \cap (D_1 \setminus s_0)$  and  $U_p^- = \{(x_1, x_2) \in \mathbb{R}^2 : \cos \alpha x_2 < \sin \alpha x_1\} \cap (D_1 \setminus s_0)$ . First, for any function  $u$  defined in a neighborhood of  $U_p^+$ , respectively  $U_p^-$ , we define the restriction

$$\mathcal{R}_p^+(u) = u|_{U_p^+}, \quad \text{respectively} \quad \mathcal{R}_p^-(u) = u|_{U_p^-}.$$

We observe that, since  $\mathcal{R}_p^\pm$  maps  $\mathcal{H}_p$  into  $H^1(U_p^\pm)$  continuously, the trace operators

$$\gamma_p^\pm : \mathcal{H}_p \longrightarrow H^{1/2}(\Gamma_p), \quad u \longmapsto \gamma_p^\pm(u) := \mathcal{R}_p^\pm(u)|_{\Gamma_p}$$

are well defined and continuous from  $\mathcal{H}_p$  to  $H^{1/2}(\Gamma_p)$ . Furthermore, by Sobolev trace inequalities and the Poincaré inequality of Lemma 3.3, it is easy to verify that the operator norm of  $\gamma_p^\pm$



is bounded uniformly with respect to  $p \in \mathbb{S}^1$ , in the sense that there exists a constant  $L > 0$  independent of  $p$  such that, recalling (19),

$$\|\gamma_p^\pm(u)\|_{H^{1/2}(\Gamma_p)} \leq L\|u\|_{\mathcal{H}_p} \quad \text{for all } u \in \mathcal{H}_p. \quad (20)$$

Clearly, for a continuous function  $u$ ,  $\gamma_p^+(u) = \gamma_p^-(u)$ .

Furthermore, let  $\nu^+ = (0, -1)$  and  $\nu^- = (0, 1)$  be the normal unit vectors to  $s_0$ , whereas

$$\nu_p^+ = (\sin \alpha, -\cos \alpha) \quad \text{and} \quad \nu_p^- = -\nu_p^+$$

be the normal unit vectors to  $\Gamma_p$ .

For every  $u \in C^1(D_1 \setminus (\Gamma_p \cup s_0))$  with  $\mathcal{R}_p^+(u) \in C^1(\overline{U_p^+} \setminus s_0)$  and  $\mathcal{R}_p^-(u) \in C^1(\overline{U_p^-} \setminus s_0)$ , we define the normal derivatives  $\frac{\partial^\pm u}{\partial \nu_p^\pm}$  on  $\Gamma_p$  respectively as

$$\frac{\partial^+ u}{\partial \nu_p^+} := \nabla \mathcal{R}_p^+(u) \cdot \nu_p^+ \Big|_{\Gamma_p}, \quad \text{and} \quad \frac{\partial^- u}{\partial \nu_p^-} := \nabla \mathcal{R}_p^-(u) \cdot \nu_p^- \Big|_{\Gamma_p}.$$

Analogous definitions hold for normal derivatives on  $s_0$  (which will be denoted just as  $\frac{\partial^\pm u}{\partial \nu^\pm}$ ).

For  $p \neq e$ , where  $e = (1, 0)$ , we consider the minimization problem for the functional  $J_p : \mathcal{H}_p \rightarrow \mathbb{R}$  defined as

$$\begin{aligned} J_p(u) &= \frac{1}{2} \int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla u|^2 dx + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(u) ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(u) ds \\ &= \frac{1}{2} \int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla u|^2 dx + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} (\gamma_p^+(u) - \gamma_p^-(u)) ds \end{aligned} \quad (21)$$

on the set

$$\mathcal{K}_p := \{u \in \mathcal{H}_p : \gamma_p^+(u + \psi) + \gamma_p^-(u + \psi) = 0\}.$$

The set  $\mathcal{K}_p$  is nonempty, convex and closed, the functional  $J_p$  is coercive (see (34)), so that the problem admits a unique minimum  $w_p \in \mathcal{K}_p$  which is a weak solution to the problem

$$\begin{cases} -\Delta w_p = 0, & \text{in } \mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}, \\ \gamma^+(w_p) + \gamma^-(w_p) = 0, & \text{on } s_0, \\ \gamma_p^+(w_p + \psi) + \gamma_p^-(w_p + \psi) = 0, & \text{on } \Gamma_p, \\ \frac{\partial^+ w_p}{\partial \nu^+} = \frac{\partial^- w_p}{\partial \nu^-}, & \text{on } s_0, \\ \frac{\partial^+(w_p + \psi)}{\partial \nu_p^+} = \frac{\partial^-(w_p + \psi)}{\partial \nu_p^-}, & \text{on } \Gamma_p. \end{cases} \quad (22)$$

**Remark 3.4.** We note that the trivial function is not a solution to the problem (22), since the two jump conditions for the solution and its normal derivative on  $\Gamma_p$  cannot be satisfied simultaneously by the trivial function if  $p \neq e$ . Hence  $w_p \not\equiv 0$  for all  $p \neq e$ .

One can easily see that the function  $e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} (w_p + \psi)$  satisfies (15) and (16), hence by the uniqueness stated in Proposition 2.1 we conclude that necessarily

$$\Psi_p = e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} (w_p + \psi). \quad (23)$$

On the other hand, for  $p = e$ , we consider the function  $w_k \in \mathcal{D}_s^{1,2}(\mathbb{R}_+^2)$  defined as the unique minimizer in (6). The function  $w_e$  defined as

$$w_e(x_1, x_2) = \begin{cases} w_k(x_1, x_2), & \text{if } x_2 \geq 0, \\ w_k(x_1, -x_2), & \text{if } x_2 \leq 0, \end{cases} \quad (24)$$

satisfies

$$w_e \in \mathcal{H}_e$$

and

$$\begin{cases} -\Delta(w_e + \psi) = 0, & \text{in } \mathbb{R}^2 \setminus s, \\ \gamma^+(w_e) + \gamma^-(w_e) = 0, & \text{on } s, \\ \frac{\partial^+ w_e}{\partial \nu^+} = \frac{\partial^- w_e}{\partial \nu^-}, & \text{on } s, \end{cases} \quad (25)$$

where  $s = \{(x_1, 0) : x_1 \geq 1\}$  and  $\mathcal{H}_e$  is defined as the completion of

$$\mathcal{D}_e = \{u \in H^1(\mathbb{R}^2 \setminus s) : \gamma^+(u) + \gamma^-(u) = 0 \text{ on } s \text{ and } u = 0 \text{ in neighborhoods of } 0 \text{ and } \infty\}$$

with respect to the Dirichlet norm  $\|\nabla u\|_{L^2(\mathbb{R}^2 \setminus s)}$ . One can easily see that the function  $e^{\frac{i}{2}\theta_e}(w_e + \psi)$  satisfies (15) and (16) with  $p = e$  (notice that  $\theta_0^e = \theta_0$ ), hence by the uniqueness stated in Proposition 2.1 we conclude that necessarily

$$\Psi_e = e^{\frac{i}{2}\theta_e}(w_e + \psi). \quad (26)$$

In [2, Proposition 14] it was proved that

$$\lim_{a=|a|p \rightarrow 0} \frac{\lambda_0 - \lambda_a}{|a|^k} = |\beta_2|^2 k \int_0^{2\pi} w_p(\cos t, \sin t) \sin\left(\frac{k}{2}t\right) dt,$$

which, combined with (5), yields

$$-4\mathfrak{m}_k \cos(k\alpha) = k \int_0^{2\pi} w_p(\cos t, \sin t) \sin\left(\frac{k}{2}t\right) dt. \quad (27)$$

The right hand side of (27) can be related to  $J_p(w_p)$  as follows.

**Lemma 3.5.** *For every  $p \neq e$*

$$\int_0^{2\pi} w_p(\cos t, \sin t) \sin\left(\frac{k}{2}t\right) dt = -\frac{2}{k} J_p(w_p).$$

*Proof.* Throughout this proof, let us denote

$$\omega_p(r) := \int_0^{2\pi} w_p(r \cos t, r \sin t) \sin\left(\frac{k}{2}t\right) dt.$$

Then we have to prove that  $k\omega_p(1) = -2J_p(w_p)$ . Since  $-\Delta w_p = 0$  in  $\mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}$ ,  $\gamma^+(w_p) + \gamma^-(w_p) = 0$  on  $s_0$ , and  $\frac{\partial^+ w_p}{\partial \nu^+} = \frac{\partial^- w_p}{\partial \nu^-}$  on  $s_0$ , by direct calculations  $\omega_p$  satisfies

$$-(r^{1+k}(r^{-k/2}\omega_p(r)))' = 0, \quad \text{in } (1, +\infty).$$

Hence there exists a constant  $C \in \mathbb{R}$  such that

$$r^{-k/2}\omega_p(r) = \omega_p(1) + \frac{C}{k} \left(1 - \frac{1}{r^k}\right), \quad \text{for all } r \geq 1.$$

From (23) and Proposition 2.1, it follows that  $\omega_p(r) = O(r^{-1/2})$  as  $r \rightarrow +\infty$ . Hence, letting  $r \rightarrow +\infty$  in the previous relation, we find  $C = -k\omega_p(1)$ , so that  $\omega_p(r) = \omega_p(1)r^{-k/2}$  for all  $r \geq 1$ . By taking the derivative in this relation and in the definition of  $\omega_p$ , we obtain

$$-\frac{k}{2}\omega_p(1) = \int_{\partial D_1} \frac{\partial w_p}{\partial \nu} \psi ds.$$

Multiplying equation (22) by  $\psi$  and integrating by parts over  $D_1 \setminus \{s_0 \cup \Gamma_p\}$ , we obtain

$$\begin{aligned} \int_{D_1 \setminus \{s_0 \cup \Gamma_p\}} \nabla w_p \cdot \nabla \psi \, dx &= \int_{\partial D_1} \frac{\partial w_p}{\partial \nu} \psi \, ds + \int_{\Gamma_p} \left( \frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-} \right) \psi \, ds \\ &= -\frac{k}{2} \omega_p(1) + \int_{\Gamma_p} \left( \frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-} \right) \psi \, ds. \end{aligned} \quad (28)$$

Testing the equation  $-\Delta \psi = 0$  by  $w_p$  and integrating by parts in  $D_1 \setminus \{s_0 \cup \Gamma_p\}$ , we arrive at

$$\begin{aligned} \int_{D_1 \setminus \{s_0 \cup \Gamma_p\}} \nabla w_p \cdot \nabla \psi \, dx &= \int_{\partial D_1} \frac{\partial \psi}{\partial \nu} w_p \, ds + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} (\gamma_p^+(w_p) - \gamma_p^-(w_p)) \, ds \\ &= \frac{k}{2} \omega_p(1) + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} (\gamma_p^+(w_p) - \gamma_p^-(w_p)) \, ds, \end{aligned} \quad (29)$$

where in the last step we used the fact that  $\frac{\partial \psi}{\partial \nu} = \frac{k}{2} \psi$  on  $\partial D_1$ . Combining (28) and (29), we obtain

$$k \omega_p(1) = \int_{\Gamma_p} \left( \frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-} \right) \psi \, ds - \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} (\gamma_p^+(w_p) - \gamma_p^-(w_p)) \, ds. \quad (30)$$

On the other hand, multiplying (22) by  $w_p$  and integrating by parts over  $\mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}$ , we obtain

$$\int_{\mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}} |\nabla w_p|^2 \, dx = \int_{\Gamma_p} \frac{\partial^+ w_p}{\partial \nu_p^+} \gamma_p^+(w_p) \, ds + \int_{\Gamma_p} \frac{\partial^- w_p}{\partial \nu_p^-} \gamma_p^-(w_p) \, ds.$$

At the same time, recalling the definition of  $J_p$  (21) and taking into account the latter equation we have

$$\begin{aligned} 2J_p(w_p) &= \int_{\mathbb{R}^2 \setminus \{s_0 \cup \Gamma_p\}} |\nabla w_p|^2 \, dx + 2 \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p) \, ds + 2 \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p) \, ds \\ &= \int_{\Gamma_p} \frac{\partial^+ w_p}{\partial \nu_p^+} \gamma_p^+(w_p) \, ds + \int_{\Gamma_p} \frac{\partial^- w_p}{\partial \nu_p^-} \gamma_p^-(w_p) \, ds + 2 \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p) \, ds + 2 \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p) \, ds \\ &= \int_{\Gamma_p} \frac{\partial^+(w_p + \psi)}{\partial \nu_p^+} \gamma_p^+(w_p) \, ds + \int_{\Gamma_p} \frac{\partial^-(w_p + \psi)}{\partial \nu_p^-} \gamma_p^-(w_p) \, ds \\ &\quad + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p) \, ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p) \, ds \\ &= \int_{\Gamma_p} \frac{\partial^+(w_p + \psi)}{\partial \nu_p^+} \gamma_p^+(w_p + \psi) \, ds + \int_{\Gamma_p} \frac{\partial^-(w_p + \psi)}{\partial \nu_p^-} \gamma_p^-(w_p + \psi) \, ds \\ &\quad + \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p) \, ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p) \, ds \\ &\quad - \int_{\Gamma_p} \frac{\partial^+(w_p + \psi)}{\partial \nu_p^+} \gamma_p^+(\psi) \, ds - \int_{\Gamma_p} \frac{\partial^-(w_p + \psi)}{\partial \nu_p^-} \gamma_p^-(\psi) \, ds \end{aligned}$$

from which the thesis follows by comparison with (30) recalling that in the last equivalence the first term is zero by (22) and  $\psi$  is regular on  $\Gamma_p$ .  $\square$

From the fact that  $w_k$  attains the minimum in (6) and (24) it follows easily that

$$\mathbf{m}_k = \frac{1}{2} \left[ \frac{1}{2} \int_{\mathbb{R}^2 \setminus s_0} |\nabla w_e|^2 \, dx + \int_{\Gamma_e} \frac{\partial^+ \psi}{\partial \nu^+} \gamma^+(w_e) \, ds + \int_{\Gamma_e} \frac{\partial^- \psi}{\partial \nu^-} \gamma^-(w_e) \, ds \right]. \quad (31)$$

Combining (27), Lemma 3.5, and (31) we conclude that, for every  $p = (\cos \alpha, \sin \alpha) \in \mathbb{S}^1 \setminus \{e\}$ ,

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla w_p|^2 dx &+ \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p) ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p) ds \\ &= \cos(k\alpha) \left[ \frac{1}{2} \int_{\mathbb{R}^2 \setminus s_0} |\nabla w_e|^2 dx + \int_{\Gamma_e} \frac{\partial^+ \psi}{\partial \nu^+} \gamma^+(w_e) ds + \int_{\Gamma_e} \frac{\partial^- \psi}{\partial \nu^-} \gamma^-(w_e) ds \right]. \end{aligned} \quad (32)$$

**Lemma 3.6.** (i) *There exists  $C > 0$  (independent of  $p \in \mathbb{S}^1$ ) such that, for all  $p \in \mathbb{S}^1$ ,*

$$\int_{\mathbb{R}^2 \setminus \Gamma_p} |(i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla \psi|^2 dx \leq C. \quad (33)$$

(ii) *If  $p_n, p \in \mathbb{S}^1$  and  $p_n \rightarrow p$  in  $\mathbb{S}^1$ , then  $\Psi_{p_n} \rightarrow \Psi_p$  weakly in  $H^1(D_R, \mathbb{C})$  for every  $R > 1$ , a.e., and in  $C_{\text{loc}}^{0,\alpha}(\mathbb{R}^2 \setminus \{p\})$ .*

*Proof.* Let us fix  $q > 2$ . From the continuity of the embedding  $H^{1/2}(\Gamma_p) \hookrightarrow L^q(\Gamma_p)$  and (20), we have that there exists some  $\text{const} > 0$  independent of  $p \in \mathbb{S}^1$  such that, for all  $u \in \mathcal{H}_p$ ,

$$\begin{aligned} \left| \int_{\Gamma_p} \frac{\partial^\pm \psi}{\partial \nu_p^\pm} \gamma_p^\pm(u) ds \right| &= \left| \frac{k}{2} \cos\left(\frac{k}{2}\alpha\right) \int_{\Gamma_p} |x|^{\frac{k}{2}-1} \gamma_p^\pm(u) ds \right| \\ &\leq \frac{k}{2} \| |x|^{\frac{k}{2}-1} \|_{L^{q'}(\Gamma_p)} \|\gamma_p^\pm(u)\|_{L^q(\Gamma_p)} \leq \text{const} \|\gamma_p^\pm(u)\|_{H^{1/2}(\Gamma_p)} \leq \text{const} L \|u\|_{\mathcal{H}_p} \end{aligned}$$

and then, from the elementary inequality  $ab \leq \frac{a^2}{4\varepsilon} + \varepsilon b^2$ , we deduce that, for every  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon > 0$  (depending on  $\varepsilon$  but independent of  $p$ ) such that, for every  $u \in \mathcal{H}_p$ ,

$$\left| \int_{\Gamma_p} \frac{\partial^\pm \psi}{\partial \nu_p^\pm} \gamma_p^\pm(u) ds \right| \leq \varepsilon \|u\|_{\mathcal{H}_p}^2 + C_\varepsilon. \quad (34)$$

From (34) and the fact that the right hand side of (32) is bounded uniformly with respect to  $p \in \mathbb{S}^1$ , we deduce that for any  $p = (\cos \alpha, \sin \alpha) \in \mathbb{S}^1$

$$\int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla w_p|^2 \leq M \quad (35)$$

for a constant  $M > 0$  independent of  $p$ . Replacing (23) ((26) for  $p = e$ ) into (35) we obtain (33).

We have that (33) together with the Hardy-type inequality of [12] implies that  $\{\Psi_p\}_{p \in \mathbb{S}^1}$  is bounded in  $H^1(D_R)$  and  $\{A_p \Psi_p\}_{p \in \mathbb{S}^1}$  is bounded in  $L^2(D_R)$  for every  $R > 1$ . Hence, by a diagonal process, for every sequence  $p_n \rightarrow p$  in  $\mathbb{S}^1$ , there exist a subsequence (still denoted as  $p_n$ ) and some  $\Psi \in H_{\text{loc}}^1(\mathbb{R}^2)$  such that  $\Psi_{p_n}$  converges to  $\Psi$  weakly in  $H^1(D_R)$  and a.e. and  $A_{p_n} \Psi_{p_n}$  converges to  $A_p \Psi$  weakly in  $L^2(D_R)$  for every  $R > 1$ . In particular this implies that  $\Psi \in H_{\text{loc}}^{1,p}(\mathbb{R}^2, \mathbb{C})$ . Passing to the limit in the equation  $(i\nabla + A_{p_n})^2 \Psi_{p_n} = 0$ , we obtain that  $(i\nabla + A_p)^2 \Psi = 0$ . Furthermore, by weak convergences  $\nabla \Psi_{p_n} \rightharpoonup \nabla \Psi$ ,  $A_{p_n} \Psi_{p_n} \rightharpoonup A_p \Psi$  in  $L^2(D_R)$  and (33), we have that, for every  $R > 1$ ,

$$\begin{aligned} \int_{D_R \setminus D_1} |(i\nabla + A_p)\Psi - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla \psi|^2 dx \\ \leq \liminf_{n \rightarrow \infty} \int_{D_R \setminus D_1} |(i\nabla + A_{p_n})\Psi_{p_n} - e^{\frac{i}{2}(\theta_{p_n} - \theta_0^{p_n})} e^{\frac{i}{2}\theta_0} i\nabla \psi|^2 dx \leq C \end{aligned}$$

and, since  $C$  is independent of  $R$ ,  $\int_{\mathbb{R}^2 \setminus D_1} |(i\nabla + A_p)\Psi - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla \psi|^2 dx \leq C$ . By the uniqueness stated in Proposition 2.1 we conclude that necessarily  $\Psi = \Psi_p$ . Since the limit  $\Psi$  depends neither on the sequence  $p_n$  nor on the subsequence, we obtain statement (ii). The convergence in  $C_{\text{loc}}^{0,\alpha}(\mathbb{R}^2 \setminus \{p\})$  follows by classical elliptic regularity theory.  $\square$

**Lemma 3.7.** *For every  $p \in \mathbb{S}^1$ , let  $f_p : [0, 1] \rightarrow \mathbb{C}$ ,  $f_p(r) = \Psi_p(rp)$ . If  $p_n, p \in \mathbb{S}^1$  and  $p_n \rightarrow p$ , then  $f_{p_n} \rightharpoonup f_p$  weakly in  $L^q(0, 1)$  for all  $q > 2$ .*

*Proof.* If  $p_n \rightarrow p$  in  $\mathbb{S}^1$ , then the  $C_{\text{loc}}^{0,\alpha}(\mathbb{R}^2 \setminus \{p\})$ -convergence stated in Lemma 3.6 implies that  $f_{p_n} \rightarrow f_p$  a.e. in  $(0, 1)$ . Furthermore, from the continuity of the embedding  $H^{1/2}(\Gamma_p) \hookrightarrow L^q(\Gamma_p)$  and boundedness of  $\{\Psi_p\}_{p \in \mathbb{S}^1}$  in  $H^1(D_1, \mathbb{C})$ , we have that

$$\|f_{p_n}\|_{L^q(0,1)} = \left( \int_{\Gamma_{p_n}} |\Psi_{p_n}|^q ds \right)^{1/q} \leq \text{const} \|\Psi_{p_n}\|_{H^{1/2}(\Gamma_{p_n})} \leq \text{const} \|\Psi_{p_n}\|_{H^1(D_1)} \leq \text{const}$$

for positive  $\text{const} > 0$  independent of  $n$ . Then, along a subsequence,  $f_{p_n}$  converges weakly in  $L^q(0, 1)$  to some limit which necessarily coincides with  $f_p$  by a.e. convergence (then the convergence holds not only along the subsequence).  $\square$

**Proposition 3.8.** *For  $\alpha \in \mathbb{R}$ , let  $p = (\cos \alpha, \sin \alpha)$ . Let  $w_p$  be the unique solution to problem (22) ((25) if  $p = e$ ). Then the function*

$$\alpha \mapsto \frac{1}{2} \int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla w_{(\cos \alpha, \sin \alpha)}(x)|^2 dx \quad (36)$$

is continuous, even and periodic with period  $\frac{2\pi}{k}$ .

*Proof.* In view of (32), to prove the continuity of the map in (36) it is enough to show that the function

$$G : \mathbb{S}^1 \rightarrow \mathbb{R}, \quad G(p) = \begin{cases} \int_{\Gamma_p} \frac{\partial^+ \psi}{\partial \nu_p^+} \gamma_p^+(w_p) ds + \int_{\Gamma_p} \frac{\partial^- \psi}{\partial \nu_p^-} \gamma_p^-(w_p) ds, & \text{if } p \neq e, \\ \int_{\Gamma_e} \frac{\partial^+ \psi}{\partial \nu^+} \gamma^+(w_e) ds + \int_{\Gamma_e} \frac{\partial^- \psi}{\partial \nu^-} \gamma^-(w_e) ds, & \text{if } p = e, \end{cases}$$

is continuous. In view of (23) and (26),  $G$  can be written also as

$$G(p) = \begin{cases} kie^{-\frac{i}{2}\theta_0(p)} \cos(\frac{k}{2}\theta_0(p)) \int_0^1 r^{\frac{k}{2}-1} f_p(r) dr, & \text{if } p \neq e, \\ ki \int_0^1 r^{\frac{k}{2}-1} f_e(r) dr, & \text{if } p = e, \end{cases}$$

so that, to prove the continuity of  $G$  it is enough to show that the function  $p \mapsto \int_0^1 r^{\frac{k}{2}-1} f_p(r) dr$  is continuous on  $\mathbb{S}^1$  and this follows from Lemma 3.7 and the fact that  $r^{\frac{k}{2}-1}$  is in  $L^t(0, 1)$  for all  $1 < t < 2$ .

To the last part of the proof, following closely [2, Lemma 15], we introduce the two transformations  $\mathcal{R}_1, \mathcal{R}_2$  acting on a general point

$$x = (x_1, x_2) = (r \cos t, r \sin t), \quad r > 0, \quad t \in [0, 2\pi),$$

as

$$\mathcal{R}_1(x) = \mathcal{R}_1(x_1, x_2) = M_k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad M_k = \begin{pmatrix} \cos \frac{2\pi}{k} & -\sin \frac{2\pi}{k} \\ \sin \frac{2\pi}{k} & \cos \frac{2\pi}{k} \end{pmatrix}$$

i.e.

$$\mathcal{R}_1(r \cos t, r \sin t) = \left( r \cos\left(t + \frac{2\pi}{k}\right), r \sin\left(t + \frac{2\pi}{k}\right) \right),$$

and

$$\mathcal{R}_2(x) = \mathcal{R}_2(x_1, x_2) = (x_1, -x_2),$$

i.e.

$$\mathcal{R}_2(r \cos t, r \sin t) = (r \cos(2\pi - t), r \sin(2\pi - t)).$$

The transformation  $\mathcal{R}_1$  is a rotation of  $\frac{2\pi}{k}$  and  $\mathcal{R}_2$  is a reflexion through the  $x_1$ -axis. We note that

$$\int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla w_p|^2 = \int_{\mathbb{R}^2 \setminus \Gamma_p} \left| (i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla\psi \right|^2 dx. \quad (37)$$

From the change of variable  $x = \mathcal{R}_1(y)$  and [2, Lemma 15, (58) and (66)] we have that

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus \Gamma_p} \left| (i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla\psi \right|^2 dx \\ = \int_{\mathbb{R}^2 \setminus \Gamma_{\mathcal{R}_1^{-1}(p)}} \left| (i\nabla + A_{\mathcal{R}_1^{-1}(p)})\Psi_{\mathcal{R}_1^{-1}(p)} - e^{\frac{i}{2}(\theta_{\mathcal{R}_1^{-1}(p)} - \theta_0^{\mathcal{R}_1^{-1}(p)} + \theta_0)} i\nabla\psi \right|^2 dy \end{aligned}$$

which, in view of (37), yields

$$\int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_{\mathcal{R}_1^{-1}(p)})} |\nabla w_{\mathcal{R}_1^{-1}(p)}|^2 = \int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla w_p|^2$$

and hence  $\frac{2\pi}{k}$ -periodicity of the map (36). On the other hand, from the change of variable  $x = \mathcal{R}_2(y)$  and [2, Lemma 15, (72)] we have that

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus \Gamma_p} \left| (i\nabla + A_p)\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} i\nabla\psi \right|^2 dx \\ = \int_{\mathbb{R}^2 \setminus \Gamma_{\mathcal{R}_2(p)}} \left| (i\nabla + A_{\mathcal{R}_2(p)})\Psi_{\mathcal{R}_2(p)} - e^{\frac{i}{2}(\theta_{\mathcal{R}_2(p)} - \theta_0^{\mathcal{R}_2(p)} + \theta_0)} i\nabla\psi \right|^2 dy \end{aligned}$$

which, in view of (37), yields

$$\int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_{\mathcal{R}_2(p)})} |\nabla w_{\mathcal{R}_2(p)}|^2 = \int_{\mathbb{R}^2 \setminus (s_0 \cup \Gamma_p)} |\nabla w_p|^2$$

and hence evenness of the map (36).  $\square$

## 4 Rate of convergence for eigenfunctions

In this section we prove a sharp estimate for the rate of convergence of eigenfunctions. The estimate of the energy variation will be derived first inside disks with radius of order  $|a|$  and later outside such disks.

### 4.1 Energy variation inside disks with radius of order $|a|$

As a straightforward corollary of the blow-up results described in section 2, we obtain the following result.

**Lemma 4.1.** *Under the same assumptions as in Theorem 2.2, we have that, for all  $p = (\cos \alpha, \sin \alpha) \in \mathbb{S}^1$  and  $R > 2$ ,*

$$\lim_{a=|a|p \rightarrow 0} \frac{1}{|a|^k} \int_{D_{R|a|}} \left| (i\nabla + A_a)\varphi_a(x) - e^{-\frac{i}{2}(\theta_0^a - \theta_a)(x)} (i\nabla + A_0)\varphi_0(x) \right|^2 dx = |\beta_2|^2 \mathcal{F}_p(R)$$

where

$$\mathcal{F}_p(R) = \int_{D_R} \left| (i\nabla + A_p)\Psi_p(x) - e^{-\frac{i}{2}(\theta_0^p - \theta_p)(x)} (i\nabla + A_0)(e^{\frac{i}{2}\theta_0}\psi)(x) \right|^2 dx.$$

*Proof.* By a change of variable we obtain that

$$\begin{aligned} \int_{D_{R|a|}} \left| (i\nabla + A_a)\varphi_a(x) - e^{-\frac{i}{2}(\theta_0^a - \theta_a)(x)} (i\nabla + A_0)\varphi_0(x) \right|^2 dx \\ = |a|^k \int_{D_R} \left| (i\nabla + A_p)\tilde{\varphi}_a(x) - e^{-\frac{i}{2}(\theta_0^p - \theta_p)(x)} (i\nabla + A_0)W_a(x) \right|^2 dx \end{aligned}$$

so that the conclusion follows from convergence (12) and Theorem 2.2.  $\square$

**Lemma 4.2.** *Let  $\mathcal{F}_p(R)$  be as in Lemma 4.1. Then*

$$\lim_{R \rightarrow +\infty} \mathcal{F}_p(R) = \mathfrak{L}_p > 0$$

where

$$\mathfrak{L}_p = \int_{\mathbb{R}^2 \setminus (\Gamma_p \cup s_0)} |\nabla w_p|^2$$

and  $w_p$  is the weak solution to the problem (22).

*Proof.* Via a change of gauge, we can write

$$\mathcal{F}_p(R) = \int_{D_R \setminus (\Gamma_p \cup s_0)} \left| e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} (i\nabla(w_p + \psi) - i\nabla\psi) \right|^2 = \int_{D_R \setminus (\Gamma_p \cup s_0)} |\nabla w_p|^2 \rightarrow \int_{\mathbb{R}^2 \setminus (\Gamma_p \cup s_0)} |\nabla w_p|^2$$

as  $R \rightarrow +\infty$ . Thanks to remark 3.4, we stress that the limit is non zero. This concludes the proof.  $\square$

## 4.2 Energy variation outside disks with radius of order $|a|$

In order to estimate the energy variation outside disks with radius  $R|a|$ , we consider the following operator:

$$\begin{aligned} F : \mathbb{C} \times H_0^{1,0}(\Omega, \mathbb{C}) &\longrightarrow \mathbb{R} \times \mathbb{R} \times (H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^* \\ (\lambda, \varphi) &\longmapsto \left( \|\varphi\|_{H_0^{1,0}(\Omega, \mathbb{C})}^2 - \lambda_0, \Im \left( \int_{\Omega} \varphi \overline{\varphi_0} dx \right), (i\nabla + A_0)^2 \varphi - \lambda \varphi \right). \end{aligned}$$

In the above definition,  $(H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^*$  is the real dual space of  $H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}) = H_0^{1,0}(\Omega, \mathbb{C})$ , which is here meant as a vector space over  $\mathbb{R}$  endowed with the norm

$$\|u\|_{H_0^{1,0}(\Omega, \mathbb{C})} = \left( \int_{\Omega} |(i\nabla + A_0)u|^2 dx \right)^{1/2},$$

and  $(i\nabla + A_0)^2 \varphi - \lambda \varphi \in (H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^*$  acts as

$$(H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^* \left\langle (i\nabla + A_0)^2 \varphi - \lambda \varphi, u \right\rangle_{H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C})} = \Re \left( \int_{\Omega} (i\nabla + A_0)\varphi \cdot \overline{(i\nabla + A_0)u} dx - \lambda \int_{\Omega} \varphi \overline{u} dx \right)$$

for all  $\varphi \in H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C})$ .

**Lemma 4.3.** *For  $\alpha \in [0, 2\pi)$ ,  $p = (\cos \alpha, \sin \alpha)$  and  $a = |a|p \in \Omega$ , let  $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$  solve (7-8) and  $\varphi_0 \in H_0^{1,0}(\Omega, \mathbb{C})$  be a solution to (3) satisfying (1), (4), and (11). Then, for all  $R > 2$ ,*

$$\|e^{\frac{i}{2}(\theta_0^a - \theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0\|_{L^2(\Omega \setminus D_{R|a|}, \mathbb{C})}^2 \leq |a|^k g(a, R)$$

where, for all  $R > 2$ ,

$$\lim_{a=|a|p \rightarrow 0} g(a, R) = g(R) \tag{38}$$

and

$$\lim_{R \rightarrow +\infty} g(R) = 0. \tag{39}$$

*Proof.* From [1, Lemma 7.1] we know that the function  $F$  is Fréchet-differentiable at  $(\lambda_0, \varphi_0)$  and its Fréchet-differential  $dF(\lambda_0, \varphi_0)$  is invertible. From the invertibility of  $dF(\lambda_0, \varphi_0)$  it follows that

$$\begin{aligned} & \left\| e^{\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_a) \varphi_a - (i\nabla + A_0) \varphi_0 \right\|_{L^2(\Omega \setminus D_{R|a|}, \mathbb{C})} \\ &= \left\| (i\nabla + A_0) (e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_a - \varphi_0) \right\|_{L^2(\Omega \setminus D_{R|a|}, \mathbb{C})} \\ &\leq |\lambda_a - \lambda_0| + \|v_{R,a} - \varphi_0\|_{H_0^{1,0}(\Omega, \mathbb{C})} \\ &\leq \|(dF(\lambda_0, \varphi_0))^{-1}\|_{\mathcal{L}(\mathbb{R} \times \mathbb{R} \times (H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^*, \mathbb{C} \times H_0^{1,0}(\Omega, \mathbb{C}))} \|F(\lambda_a, v_{R,a})\|_{\mathbb{R} \times \mathbb{R} \times (H_{0,\mathbb{R}}^{1,0}(\Omega))^*} (1 + o(1)) \end{aligned}$$

as  $|a| \rightarrow 0^+$ . We have that

$$F(\lambda_a, v_{R,a}) = (\alpha_a, \beta_a, w_a)$$

where

$$\begin{aligned} \alpha_a &= \|v_{R,a}\|_{H_0^{1,0}(\Omega, \mathbb{C})}^2 - \lambda_0 \in \mathbb{R}, \\ \beta_a &= \Im \left( \int_{\Omega} v_{R,a} \overline{\varphi_0} dx \right) \in \mathbb{R}, \\ w_a &= (i\nabla + A_0)^2 v_{R,a} - \lambda_a v_{R,a} \in (H_{0,\mathbb{R}}^{1,0}(\Omega))^*. \end{aligned}$$

We mention that in [1, 2], the norm of  $\|F(\lambda_a, v_{R,a})\|_{\mathbb{R} \times \mathbb{R} \times (H_{0,\mathbb{R}}^{1,0}(\Omega))^*}$  was estimated before proving the blow-up results recalled in Theorem 2.2 and Lemma 2.3 (actually some preliminary estimates of  $F(\lambda_a, v_{R,a})$  were carried out to obtain an energy control in terms of an implicit normalization needed to prove the blow-up results). Here we are going to exploit the sharp blow-up results Theorem 2.2 and Lemma 2.3 to improve the preliminary estimates in [1, 2]. From (5), Theorem 2.2 and Lemma 2.3 we have that

$$\begin{aligned} \alpha_a &= \left( \int_{D_{R|a|}} |(i\nabla + A_0) v_{R,a}^{int}|^2 dx - \int_{D_{R|a|}} |(i\nabla + A_a) \varphi_a|^2 dx \right) + (\lambda_a - \lambda_0) \\ &= |a|^k \left( \int_{D_R} |(i\nabla + A_0) Z_a^R|^2 dx - \int_{D_R} |(i\nabla + A_p) \tilde{\varphi}_a|^2 dx \right) + (\lambda_a - \lambda_0) = O(|a|^k), \end{aligned}$$

as  $|a| \rightarrow 0^+$ . The normalization condition for the phase in (8) together with the blow-up results (12), Theorem 2.2 and Lemma 2.3 yield

$$\begin{aligned} \beta_a &= \Im \left( \int_{D_{R|a|}} v_{R,a}^{int} \overline{\varphi_0} dx - \int_{D_{R|a|}} e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_a \overline{\varphi_0} dx + \int_{\Omega} e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_a \overline{\varphi_0} dx \right) \\ &= \Im \left( |a|^{k+2} \int_{D_R} Z_a^R \overline{W_a} dx - |a|^{k+2} \int_{D_R} e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a \overline{W_a} dx \right) = O(|a|^{k+2}) \quad \text{as } |a| \rightarrow 0^+. \end{aligned}$$

Let  $\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})$  be the functional space defined in (14). For every  $a \in \Omega$ , we define the map

$$\mathcal{T}_a : \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}) \rightarrow \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}), \quad \mathcal{T}_a \varphi(x) = \varphi(|a|x).$$

It is easy to verify that  $\mathcal{T}_a$  is an isometry of  $\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})$ .

Since  $H_0^{1,0}(\Omega, \mathbb{C})$  can be thought as continuously embedded into  $\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})$  by trivial extension outside  $\Omega$  and  $\|u\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})} = \|u\|_{H_0^{1,0}(\Omega, \mathbb{C})}$  for every  $u \in H_0^{1,0}(\Omega, \mathbb{C})$ , we have that

$$\begin{aligned} & \|w_a\|_{(H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^*} \\ &= \sup_{\substack{\varphi \in H_0^{1,0}(\Omega, \mathbb{C}) \\ \|\varphi\|_{H_0^{1,0}(\Omega, \mathbb{C})} = 1}} \left| \Re \left( \int_{\Omega} (i\nabla + A_0) v_{R,a} \cdot \overline{(i\nabla + A_0) \varphi} dx - \lambda_a \int_{\Omega} v_{R,a} \overline{\varphi} dx \right) \right| \\ &\leq \sup_{\substack{\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}) \\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})} = 1}} \left| \Re \left( \int_{\Omega} (i\nabla + A_0) v_{R,a} \cdot \overline{(i\nabla + A_0) \varphi} dx - \lambda_a \int_{\Omega} v_{R,a} \overline{\varphi} dx \right) \right|. \end{aligned} \quad (40)$$



For every  $\varphi \in H_0^{1,0}(\Omega, \mathbb{C})$  we have that

$$\begin{aligned}
& \int_{\Omega} (i\nabla + A_0)v_{R,a} \cdot \overline{(i\nabla + A_0)\varphi} dx - \lambda_a \int_{\Omega} v_{R,a} \overline{\varphi} dx \\
&= \int_{\Omega \setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_a)\varphi_a \cdot \overline{(i\nabla + A_0)\varphi} dx - \lambda_a \int_{\Omega \setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_a \overline{\varphi} dx \\
&\quad + \int_{D_{R|a|}} (i\nabla + A_0)v_{R,a} \cdot \overline{(i\nabla + A_0)\varphi} dx - \lambda_a \int_{D_{R|a|}} v_{R,a} \overline{\varphi} dx.
\end{aligned} \tag{41}$$

From scaling and integration by parts

$$\begin{aligned}
& \int_{\Omega \setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_a)\varphi_a \cdot \overline{(i\nabla + A_0)\varphi} dx - \lambda_a \int_{\Omega \setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0^a - \theta_a)} \varphi_a \overline{\varphi} dx \\
&= |a|^{\frac{k}{2}} \left( \int_{\frac{\Omega}{|a|} \setminus D_R} e^{\frac{i}{2}(\theta_0^p - \theta_p)} (i\nabla + A_p)\tilde{\varphi}_a \cdot \overline{(i\nabla + A_0)(\mathcal{T}_a\varphi)} dx - \lambda_a |a|^2 \int_{\frac{\Omega}{|a|} \setminus D_R} e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a \overline{\mathcal{T}_a\varphi} dx \right) \\
&= |a|^{\frac{k}{2}} \left( \int_{\frac{\Omega}{|a|} \setminus D_R} (i\nabla + A_p)\tilde{\varphi}_a \cdot \overline{(i\nabla + A_p)(e^{-\frac{i}{2}(\theta_0^p - \theta_p)} \mathcal{T}_a\varphi)} dx - \lambda_a |a|^2 \int_{\frac{\Omega}{|a|} \setminus D_R} \tilde{\varphi}_a \overline{e^{-\frac{i}{2}(\theta_0^p - \theta_p)} \mathcal{T}_a\varphi} dx \right) \\
&= |a|^{\frac{k}{2}} i \int_{\partial D_R} \overline{\mathcal{T}_a\varphi} e^{\frac{i}{2}(\theta_0^p - \theta_p)} (i\nabla + A_p)\tilde{\varphi}_a \cdot \nu d\sigma \\
&= |a|^{\frac{k}{2}} i \int_{\partial D_R} \overline{\mathcal{T}_a\varphi} (i\nabla + A_0)(e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a) \cdot \nu d\sigma
\end{aligned} \tag{42}$$

being  $\nu = \frac{x}{|x|}$  the outer unit vector. In a similar way we have that

$$\begin{aligned}
& \int_{D_{R|a|}} (i\nabla + A_0)v_{R,a} \cdot \overline{(i\nabla + A_0)\varphi} dx - \lambda_a \int_{D_{R|a|}} v_{R,a} \overline{\varphi} dx \\
&= |a|^{\frac{k}{2}} \left( \int_{D_R} (i\nabla + A_0)Z_a^R \cdot \overline{(i\nabla + A_0)(\mathcal{T}_a\varphi)} dx - \lambda_a |a|^2 \int_{D_R} Z_a^R \overline{\mathcal{T}_a\varphi} dx \right) \\
&= |a|^{\frac{k}{2}} \left( -i \int_{\partial D_R} (i\nabla + A_0)Z_a^R \cdot \nu \overline{\mathcal{T}_a\varphi} d\sigma - \lambda_a |a|^2 \int_{D_R} Z_a^R \overline{\mathcal{T}_a\varphi} dx \right).
\end{aligned} \tag{43}$$

Combining (40), (41), (42), (43), and recalling that  $\mathcal{T}_a$  is an isometry of  $\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})$ , we obtain

that

$$\begin{aligned}
& |a|^{-\frac{k}{2}} \|w_a\|_{(H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^*} \\
& \leq \sup_{\substack{\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}) \\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})} = 1}} \left| i \int_{\partial D_R} (i\nabla + A_0) \left( e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \overline{\mathcal{T}_a \varphi} d\sigma - \lambda_a |a|^2 \int_{D_R} Z_a^R \overline{\mathcal{T}_a \varphi} dx \right| \\
& = \sup_{\substack{\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}) \\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})} = 1}} \left| i \int_{\partial D_R} (i\nabla + A_0) \left( e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \overline{\varphi} d\sigma - \lambda_a |a|^2 \int_{D_R} Z_a^R \overline{\varphi} dx \right| \\
& \leq \sup_{\substack{\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}) \\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})} = 1}} \left| \int_{\partial D_R} (i\nabla + A_0) \left( e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \overline{\varphi} d\sigma \right| \\
& \quad + \lambda_a |a|^2 \sup_{\substack{\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}) \\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})} = 1}} \left| \int_{D_R} Z_a^R \overline{\varphi} dx \right| \\
& = \sup_{\substack{\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}) \\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})} = 1}} \left| \int_{\partial D_R} (i\nabla + A_0) \left( e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \overline{\varphi} d\sigma \right| + |a|^2 O(\|Z_a^R\|_{L^2(D_R, \mathbb{C})}).
\end{aligned}$$

From Theorem 2.2 and Lemma 2.3 it follows that

$$(i\nabla + A_0) \left( e^{\frac{i}{2}(\theta_0^p - \theta_p)} \tilde{\varphi}_a - Z_a^R \right) \cdot \nu \rightarrow \beta_2 (i\nabla + A_0) \left( e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p - z_{p,R} \right) \cdot \nu \quad \text{in } H^{-1/2}(\partial D_R)$$

as  $a = |a|p \rightarrow 0$  and

$$|a|^2 O(\|Z_a^R\|_{L^2(D_R, \mathbb{C})}) \rightarrow 0 \quad \text{as } a = |a|p \rightarrow 0.$$

Hence we conclude that

$$|a|^{-\frac{k}{2}} \|w_a\|_{(H_{0,\mathbb{R}}^{1,0}(\Omega, \mathbb{C}))^*} \leq h(a, R)$$

with

$$\lim_{a=|a|p \rightarrow 0} h(a, R) = |\beta_2| h(R)$$

being

$$h(R) = \sup_{\substack{\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C}) \\ \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})} = 1}} \left| \int_{\partial D_R} (i\nabla + A_0) \left( e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p - z_{p,R} \right) \cdot \nu \overline{\varphi} d\sigma \right|.$$

We observe that, for every  $\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})$ ,

$$\begin{aligned}
& \left| \int_{\partial D_R} (i\nabla + A_0) \left( e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p - z_{p,R} \right) \cdot \nu \overline{\varphi} d\sigma \right| \\
&= \left| \int_{\partial D_R} e^{\frac{i}{2}(\theta_0^p - \theta_p)} (i\nabla + A_p) \left( \Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi \right) \cdot \nu \overline{\varphi} d\sigma \right. \\
&\quad \left. + \int_{\partial D_R} (i\nabla + A_0) \left( e^{\frac{i}{2}\theta_0} \psi - z_{p,R} \right) \cdot \nu \overline{\varphi} d\sigma \right| \\
&= \left| -i \int_{\mathbb{R}^2 \setminus D_R} (i\nabla + A_p) \left( \Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi \right) \cdot \overline{(i\nabla + A_0)\varphi} e^{\frac{i}{2}(\theta_0^p - \theta_p)} dx \right. \\
&\quad \left. + i \int_{D_R} (i\nabla + A_0) \left( e^{\frac{i}{2}\theta_0} \psi - z_{p,R} \right) \cdot \overline{(i\nabla + A_0)\varphi} dx \right| \\
&\leq \left( \sqrt{\int_{\mathbb{R}^2 \setminus D_R} \left| (i\nabla + A_p) \left( \Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi \right) \right|^2 dx} \right. \\
&\quad \left. + \sqrt{\int_{D_R} \left| (i\nabla + A_0) \left( e^{\frac{i}{2}\theta_0} \psi - z_{p,R} \right) \right|^2 dx} \right) \|\varphi\|_{\mathcal{D}_0^{1,2}(\mathbb{R}^2, \mathbb{C})}
\end{aligned}$$

and hence

$$\begin{aligned}
h(R) &\leq \sqrt{\int_{\mathbb{R}^2 \setminus D_R} \left| (i\nabla + A_p) \left( \Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi \right) \right|^2 dx} \\
&\quad + \sqrt{\int_{D_R} \left| (i\nabla + A_0) \left( e^{\frac{i}{2}\theta_0} \psi - z_{p,R} \right) \right|^2 dx}.
\end{aligned}$$

From Proposition 2.1 it follows that  $\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^2 \setminus D_R} \left| (i\nabla + A_p) \left( \Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi \right) \right|^2 dx = 0$ .

Since  $(i\nabla + A_0)^2 (e^{\frac{i}{2}\theta_0} \psi - z_{p,R}) = 0$  in  $D_R$  and  $(e^{\frac{i}{2}\theta_0} \psi - z_{p,R})|_{\partial D_R} = e^{\frac{i}{2}\theta_0} \psi - e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p$ , if  $\eta_R$  is a smooth cut-off function satisfying

$$\eta_R \equiv 0 \text{ in } D_{R/2}, \quad \eta \equiv 1 \text{ in } \mathbb{R}^2 \setminus D_R, \quad 0 \leq \eta_R \leq 1, \quad |\nabla \eta_R| \leq \frac{4}{R} \text{ in } D_R \setminus D_{R/2},$$

from the Dirichlet Principle we can estimate

$$\begin{aligned}
& \int_{D_R} \left| (i\nabla + A_0) \left( e^{\frac{i}{2}\theta_0} \psi - z_{p,R} \right) \right|^2 dx \leq \int_{D_R} \left| (i\nabla + A_0) \left( \eta_R (e^{\frac{i}{2}\theta_0} \psi - e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p) \right) \right|^2 dx \\
&\leq 2 \int_{D_R} |\nabla \eta_R|^2 |e^{\frac{i}{2}\theta_0} \psi - e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p|^2 dx + 2 \int_{\mathbb{R}^2 \setminus D_{R/2}} \left| (i\nabla + A_0) \left( e^{\frac{i}{2}\theta_0} \psi - e^{\frac{i}{2}(\theta_0^p - \theta_p)} \Psi_p \right) \right|^2 dx \\
&\leq \frac{32}{R^2} \int_{D_R \setminus D_{R/2}} |\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi|^2 dx + 2 \int_{\mathbb{R}^2 \setminus D_{R/2}} \left| (i\nabla + A_p) \left( \Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p)} e^{\frac{i}{2}\theta_0} \psi \right) \right|^2 dx
\end{aligned}$$

which, in view of Proposition 2.1, implies that  $\lim_{R \rightarrow +\infty} \int_{D_R} \left| (i\nabla + A_0) \left( e^{\frac{i}{2}\theta_0} \psi - z_{p,R} \right) \right|^2 dx = 0$ . Therefore we can conclude that  $h(R) \rightarrow 0$  as  $R \rightarrow +\infty$ . The proof is thereby complete.  $\square$

### 4.3 Proof of Theorem 1.1

As observed in §2, it is not restrictive to assume  $\beta_1 = 0$ . Let  $\varepsilon > 0$ . From Lemma 4.2 and (39) there exists  $R_0 > 0$  sufficiently large such that

$$|\mathcal{F}_p(R_0) - \mathcal{L}_p| < \varepsilon \quad \text{and} \quad |g(R_0)| < \varepsilon.$$

From (38) and Lemma 4.1 there exists  $\delta > 0$  (depending on  $\varepsilon$  and  $R_0$ ) such that, if  $|a| < \delta$ , then

$$|g(a, R_0) - g(R_0)| < \varepsilon$$

and

$$\left| \frac{1}{|a|^k} \int_{D_{R_0|a|}} \left| (i\nabla + A_a)\varphi_a(x) - e^{-\frac{i}{2}(\theta_0^a - \theta_a)(x)} (i\nabla + A_0)\varphi_0(x) \right|^2 dx - |\beta_2|^2 \mathcal{F}_p(R_0) \right| < \varepsilon.$$

Therefore, taking into account also Lemma 4.3, we have that, for all  $a = |a|p$  with  $|a| < \delta$ ,

$$\begin{aligned} & \left| |a|^{-k} \int_{\Omega} \left| (i\nabla + A_a)\varphi_a - e^{-\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_0)\varphi_0 \right|^2 dx - |\beta_2|^2 \mathfrak{L}_p \right| \\ & \leq \left| |a|^{-k} \int_{D_{R_0|a|}} \left| (i\nabla + A_a)\varphi_a - e^{-\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_0)\varphi_0 \right|^2 dx - |\beta_2|^2 \mathcal{F}_p(R_0) \right| \\ & \quad + |a|^{-k} \int_{\Omega \setminus D_{R_0|a|}} \left| (i\nabla + A_a)\varphi_a - e^{-\frac{i}{2}(\theta_0^a - \theta_a)} (i\nabla + A_0)\varphi_0 \right|^2 dx + |\beta_2|^2 |\mathfrak{L}_p - \mathcal{F}_p(R_0)| \\ & < \varepsilon + g(a, R_0) + |\beta_2|^2 \varepsilon \leq \varepsilon + |g(a, R_0) - g(R_0)| + |g(R_0)| + |\beta_2|^2 \varepsilon = (3 + |\beta_2|^2) \varepsilon, \end{aligned}$$

thus concluding the proof of Theorem 1.1.

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